USE GAMES TO MOTIVATE YOUR CALCULUS STUDENTS

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As mathematics instructors, we are always searching for new ways to motivate our students' learning of precalculus, calculus and differential equations. How many times have we heard our students say: "When will we ever use this?" or "This is boring." Some students find mathematics dry and do not fully engage themselves in classroom discussions and activities. To help these students (and all students in general), we developed a set of computer games that they play using mathematical models. These games add a spark of fun and enjoyment for our students in their pursuit of learning mathematics.

Warm-up Exercises

In a first-year calculus course, the instructor quickly discovers students' weaknesses in using and understanding functions. Additionally, students' "working" libraries of mathematical functions are frequently limited. We designed a set of games that encourage students to develop a better understanding of functions and their properties. One of these games, *A Piecewise Plumbing System*, is described below:

> The object of this game is to find a pair of piecewise functions that trace the path of the front most pipe system going from a bathtub on the first floor to a boiler in the basement of the house. You are free to use any type of function to model each section of pipe. Game points are given for how well the piecewise model fits the pipes and bonus points are given for each additional different type of function used, for example, linear, quadratic, rational, algebraic, and so on. Find one piecewise function that models the upper portion of the pipes and a second piecewise function that models the lower portion of the pipes.

The learning objectives for this game are:

1. Find the equation of a line between two points.
2. Review polynomials, rational functions, and algebraic functions.
3. Make new functions from simple functions by shifting, stretching, or shrinking.

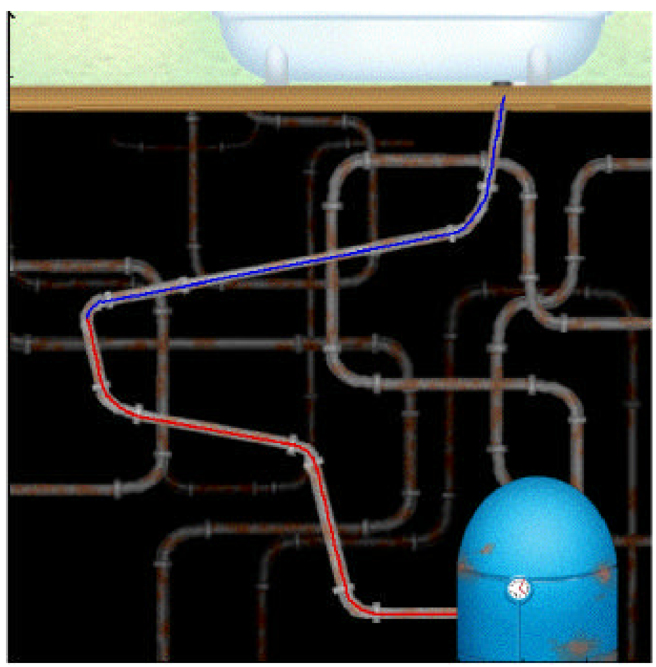

Figure 1 A Piecewise Plumbing System

4. Practice building and using piecewise functions.

The piecewise functions for the upper and lower portions of the pipes in this example are

$$f = \begin{cases} 0.6\sqrt[3]{x-1.22}+5.2 & 1.22 \le x \le 1.45 \\ 0.19(x-6.9)+6.6 & 1.45 < x \le 6.9 \\ 1.5(x-6.8)^2+6.58 & 6.9 < x \le 7.45 \\ 5.8(x-7.66)+8.5 & 7.45 < x \le 7.7 \end{cases}, \quad g = \begin{cases} -4.55(x-1.44)+4.32 & 1.22 \le x \le 1.46 \\ \dfrac{0.5}{\sqrt{x-1.2}}+3.25 & 1.46 < x \le 2 \\ -0.18(x-4.4)+3.36 & 2 < x \le 4.4 \\ -3(x-4.2)^4+3.36 & 4.4 < x \le 4.8 \\ -4.14(x-5.24)+1.12 & 4.8 < x \le 5.24 \\ -(x-5.9)^3+0.76 & 5.24 < x \le 5.74 \\ 0.76 & 5.74 < x \le 6.95 \end{cases}$$

and their graphs are shown in Figure 1.

Starting the Game

Once students have developed a good working library of functions, it becomes an easier task for them to apply the techniques of calculus. We have developed a set of games integrating data sampling, curve fitting, and calculus. The game, *Modeling Wing Aerodynamics of a Butterfly*, has the following directions:

> The Aerodynamics Research Institute is studying the flight dynamics of butterflies. As part of their study, they need an accurate estimate of the wing area of a butterfly. You have been hired as a consultant to calculate the area of the Polyfit Morpho butterfly. After studying the problem description, you have decided to fit the wing with piecewise polynomial functions and to use calculus to find the area between the polynomial functions. Your problem solving strategy is to place points along the edge of the wing and to use TEMATH's Least Squares Polynomial Fit tool (or the Interpolation tool) to find the

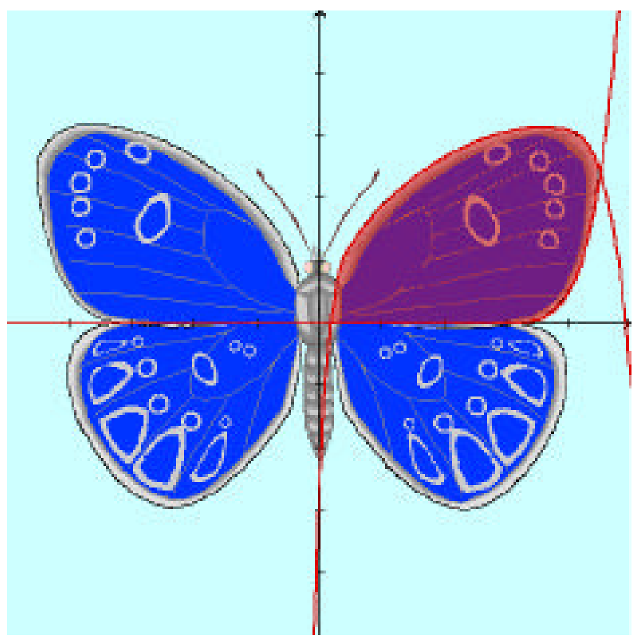

Figure 2 Finding the Area of a Butterfly's Wing

coefficients of the polynomials. Once the polynomials have been determined, you will use TEMATH's Integration of the Difference of Two Functions tool to find the area of the wing.

As is shown in Figure 2, piecewise polynomial functions provide an excellent fit to the shape of the butterfly's wing and, hence, yields an excellent estimate of the area of the wing. The total estimated area of the four wing sections of the butterfly is 37.3.

Getting into the Game

In a first-year calculus course, we introduce students to the concept of parametric equations but have little time to give them a good experience in parameterizing a variety of curves. A fun way for a student to gain experience in parameterizing curves is to play the *Parametric Racetrack* game.

You and your friend are Mathcar race drivers. Your first objective is to find the parametric equations that draw the path of your car through the center of the track for one complete revolution. Now that you have taken your practice lap, you and your friend need to find the parametric equations for the following races:

1. You and your friend race to a photo finish after one lap. Your car races on the outside of the track and your friend's car races on the inside of the track.

2. You and your friend race as described above but this time you beat your friend by a bumper. Additional points are given if you come from behind to beat your friend.

3. Your friend will mark a random point on the racetrack. You must now find the parametric equations that draw the path of your car starting at the random point and ending at the same random point after completing one revolution.

4. In the final "Ultimate Challenge" race, you and your friend both mark a random point on the racetrack. Your race begins at your friend's point and ends at the finish line. Your friend's race begins at your random point and ends at the finish line. The race must end in a photo finish.

You can use TEMATH's tracker tool to trace the path of each car in the race, watch the race in action, and observe the exciting finish of each race.

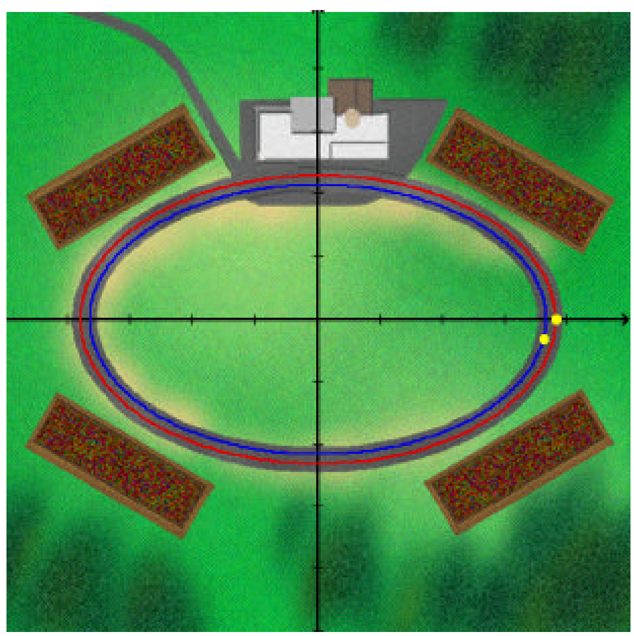

Figure 3 Parametric Racetrack

An example solution to race 2 is shown in Figure 3. The parametric equations for this race are

Outer (Red) Path	Inner (Blue) Path
$x(t) = 3.8\cos(t^2/6)$	$x(t) = 3.65\cos(t)$
$y(t) = 2.3\sin(t^2/6)$	$y(t) = 2.15\sin(t)$

Saving the Game — An Environmental Comeback

In the game *Saving the Elephants — A Polar Solution*, students use polar equations to model the shape of a lake and to calculate the area of the lake. The description of the game is given below.

> Hwange Lake in Zimbabwe, Africa is a favorite watering hole for elephants. In recent years, a plant not native to Zimbabwe has taken root in the lake and its rapid predatory growth has threatened to take over the lake and kill all native plants and amphibian life, thus killing the lake and its water source for the endangered African elephant. Your job as consultant to the Zimbabwe Natural Trust Council is to estimate the surface area of the lake. Notice that the lake is shaped like a familiar polar curve. Find the polar curve that best models the shape of the lake and use integration (or TEMATH's Integration tool) to find the area enclosed by the polar curve, and hence, obtain an estimate for the surface area of the lake. The Zimbabwe Natural Trust Council can now order the correct quantity of an environmentally safe chemical that will be used to kill the invading plant species. Assume that the units of measurement are meters.

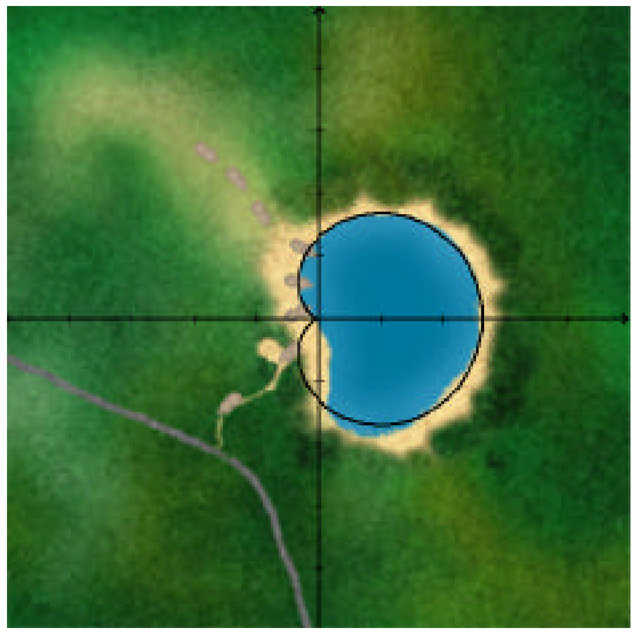

Figure 4 Saving the Elephants

An example model is shown in Figure 4. The polar equation model is the cardioid

$$r = 131 + 131\cos(t)$$

and the estimate of the area is 80,869 m^2.

Speeding to the Finish of the Game

We have also developed a set of games that are designed to help students better understand the qualitative properties of various differential equation models and their parameters. For example, in the game *The Logistic Golf Tournament*, students are asked to find the modified logistic differential equation that would sink the last shot on the 18th hole. This differential equation model

$$\frac{dy}{dt} = ky\left(\frac{y}{M} - 1\right)\left(1 - \frac{y}{N}\right)$$

has the parameters k, M, and N. In trying to model the path of the golf ball to the hole, the student needs to develop an understanding of the three parameters and how their values affect the shape of the graph of the solution to the differential equation. Building an intuition of differential equation models and their parameters is the key to performing a qualitative analysis of differential equations and their solutions. Students use TEMATH's Differential Equation Solver to enter their differential equation model and to plot its solution curve.

Figure 5 The Logistic Golf Tournament

TEMATH (Tools for Exploring Mathematics)

TEMATH (Tools for Exploring Mathematics) is a mathematics exploration environment useful for investigating a broad range of mathematical problems. It is effective for solving problems in precalculus, calculus, differential equations, linear algebra, numerical analysis, and math modeling. TEMATH contains a powerful grapher, a matrix calculator, an expression calculator, a differential equation solver, a facility for handling and manipulating data, numerical mathematical tools, and visual and dynamic exploration tools. TEMATH requires an Apple Macintosh computer (post MacPlus; PowerPC even better) running MAC OS 7.5-9.04, a 12" or larger monitor screen, 3 MB of free RAM, and 2MB of disk space (TEMATH plus files). You can download a copy of TEMATH and its documentation, application files, and games from:

www2.umassd.edu/TEMATH

Bibliography

[1] Robert Kowalczyk and Adam Hausknecht, *TEMATH - Tools for Exploring Mathematics Version 2.0.6*, 2000.

[2] Robert Kowalczyk and Adam Hausknecht, *Using TEMATH in Calculus*, 2000.

[3] The game background pictures were drawn by Josh Allan, design student, University of Massachusetts Dartmouth, 2000.

More Games

Modeling a Polynomial Roller Coaster

> 5⅞ Flags Amusement Park in Polynomial City needs to replace the safety fence along the path of their steepest descent roller coaster. The safety fence is specially made and is very expensive. For budgetary reasons, the park needs to order the exact amount of fencing. Your job as consultant is to mathematically model the path of the roller coaster using a polynomial function and to calculate the length of the path, and thus, the amount of fencing that is needed.

The learning objectives for this game are:

1. Using zeros to construct a polynomial.
2. Scaling polynomials.
3. Building an intuition on the behavior of polynomial graphs.

The length of the polynomial curve can be computed using TEMATH's Arc Length tool, or it can be an exercise in the study of arc length and numerical integration. An example solution to this game is shown in Figure 6.

The polynomial function for this fit is

$$p(x) = 0.5(x - 5.8)(x - 4)(x - 0.7)(x + 2.1)(x + 6.19)$$

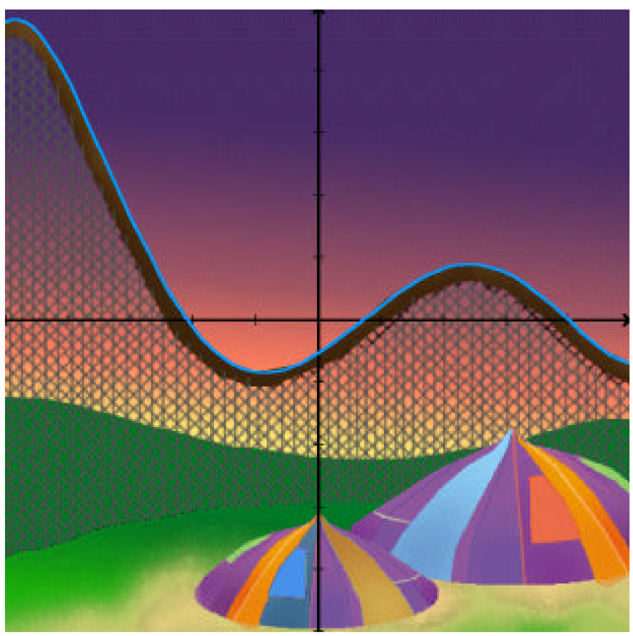

Figure 6 Modeling a Polynomial Roller Coaster

Smooth Landing on an Aircraft Carrier — A Horizontal Asymptotic Approach

You are a jet fighter pilot about to land on the USS Math Quest. You must land smoothly or considerable damage could be caused to both the jet plane and the surface of the aircraft carrier. Find a mathematical function that plots a smooth trajectory from your plane to the surface of the aircraft carrier.

The learning objectives for this game are:

1. Investigating functions with horizontal asymptotes.
2. Translation and shifting of functions.

An example solution is shown in Figure 7. The smooth landing function for this example is

$$f(x) = 3 + \frac{2}{1 + e^{x-4}}$$

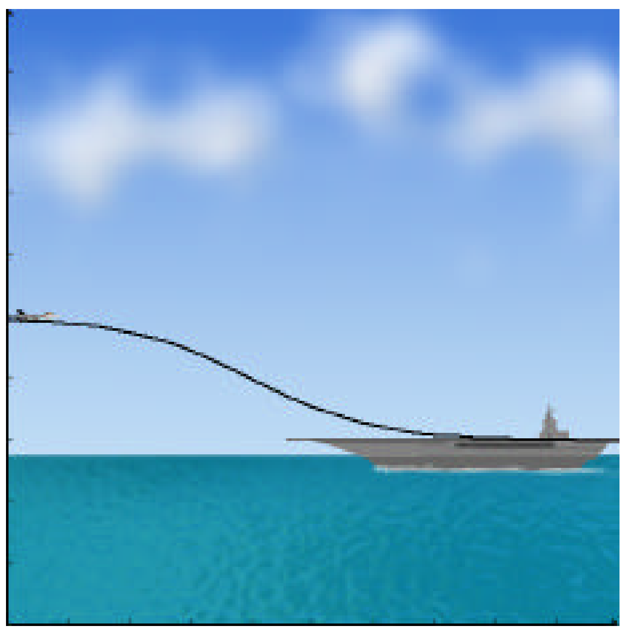

Figure 7 Smooth Landing on an Aircraft Carrier

Welcome to the Parametric Basketball Court

The objective of this game is to use the equations of motion and parametric equations to trace the trajectory of a scoring basket. Assume that distances are measured in feet and velocity is measured in ft/s. The parametric equations of motion are

$$x(t) = x_0 + v_0 \cos(a)t,\ y(t) = y_0 + v_0 \sin(a)t - (1/2)\,gt^2$$

You need to determine the values of the parameters x_0, y_0, v_0, and a that will draw a trajectory for a scoring basket. If you score a basket on your first try, you get 10 points. If you score a basket on your second try, you get 9 points. Thus, you get one less point for each try. You have ten tries to score a basket.

Try for 5 bonus points by finding the parametric equations that draw a trajectory for a bank shot off the backboard that scores a basket!

The learning objectives for this game are:

1. Developing a working knowledge of the equations of motion.
2. Studying applications of parametric equations.
3. Developing an intuition for the values of the parameters in the equations of motion.

Use TEMATH's Parametric Tracker tool to trace the trajectory and watch the action as the ball flies through the air. Example solutions are shown in Figures 8 and 9.

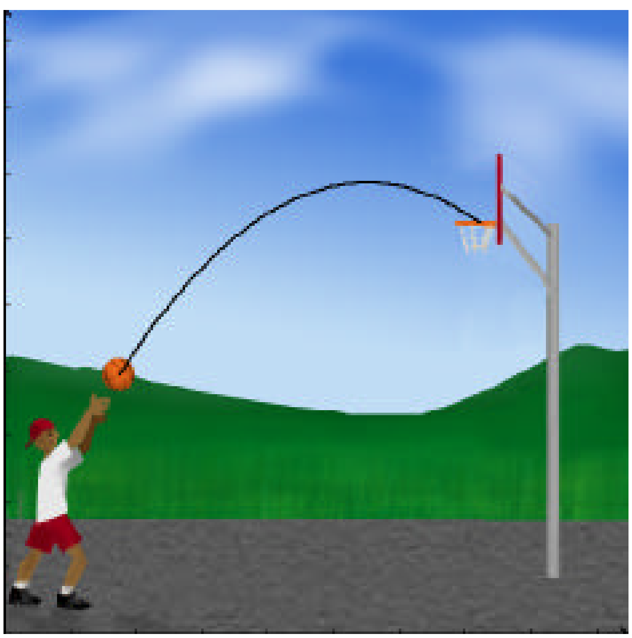

Figure 8 Shooting a Basket

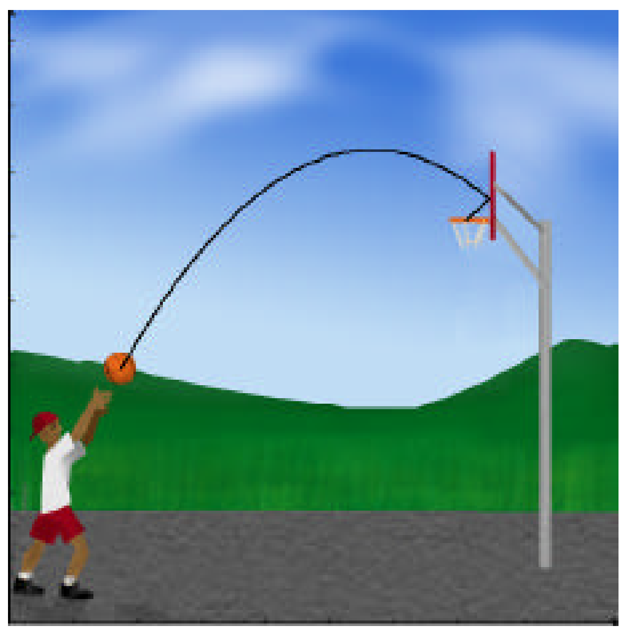

Figure 9 Scoring a Bank Shot

The parametric equations for the scoring shot in this example are

$$x(t) = 3.5 + 23\cos(1)t\,,\ y(t) = 7.9 + 23\sin(1)t - (1/2)(32)t^2$$

The parametric equations for the scoring bank shot off the backboard are

$$x(t) = \begin{cases} 3.5 + 24\cos(1.05)t & 0 \le t \le 0.96 \\ 3.5 + 24\cos(1.05)0.96 - 24\cos(1.05)(t - 0.96) & 0.96 \le t \le 1.02 \end{cases}$$

and

$$y(t) = 7.9 + 24\sin(1.05)t - (1/2)(32)t^2$$

Another version of this game is to shoot a basket over a defender's outstretched arms (see Figure 10).

The parametric equations for this example are

$$x(t) = 3.45 + 24.7\cos(1.2)t\,,\ y(t) = 8.05 + 24.7\sin(1.2)t - (1/2)(32)t^2$$

Figure 10 Scoring a Basket Over a Defender

Snowboarding Ski Jump

The objective of this game is to use the equations of motion and parametric equations to trace the jump of the snowboarder with a smooth landing on the bottom hill. Assume that distances are measured in feet and velocity is measured in ft/s.

The parametric equations of motion are

$$x(t) = x_0 + v_0 \cos(a)t,\ y(t) = y_0 + v_0 \sin(a)t - (1/2)\,gt^2$$

You need to determine the values of the parameters x_0, y_0, v_0, and a that will draw a trajectory for the jump of the snowboarder.

The learning objectives for this game are:

1. Developing a working knowledge of the equations of motion.
2. Studying applications of parametric equations.
3. Developing an intuition for the values of the parameters in the equations of motion.

Use TEMATH's Parametric Tracker tool to trace the trajectory and watch the action as the snowboarder jumps through the air.

This game can be extended to include a model for the lift the snowboarder receives as he flies through the air.

The parametric equations for this example are

$$x(t) = 87 + 57t,\ y(t) = 187 - (1/2)(32)t^2$$

Figure 11 Snowboarding Jump

The Infinity City Skate Board Park

Infinity City has built a new figure 8 skate board track in its central plaza. Youths from all over the city race their skate boards over bridges, through tunnels, and between buildings. You and your friend have entered this weekend's skate board exhibition. For your part in the exhibition, you will skate the track by starting in the center of the figure 8, skate directly into quadrant one in a clockwise direction, and make one complete lap of the track. Your friend will also start in the center of the figure 8, but will skate directly into quadrant two in a counterclockwise direction, and make two complete laps of the track. Both of you will take the same amount of time to complete your skate's.

1. Find the parametric equations that draw your path around the figure 8.
2. Find the parametric equations that draw your friend's path around the figure 8.

The learning objectives for this game are:

1. Parameterizing self-intersecting curves.
2. Building an intuition about the direction of travel along a parametric curve.

Use TEMATH's Parametric Tracker tool to trace the paths of you and your friend and watch the skate board exhibition in action.

The parametric equations for this example are

$$x(t) = 1.82\sin(t) - 0.06,\ y(t) = 1.01\sin(2t) - 0.17 \quad \text{(your path)}$$

and

$$x(t) = 1.81\sin(-2(t + 0.01)) - 0.06,\ y(t) = \sin(4(t + 0.01)) - 0.17 \quad \text{(your friend's path)}$$

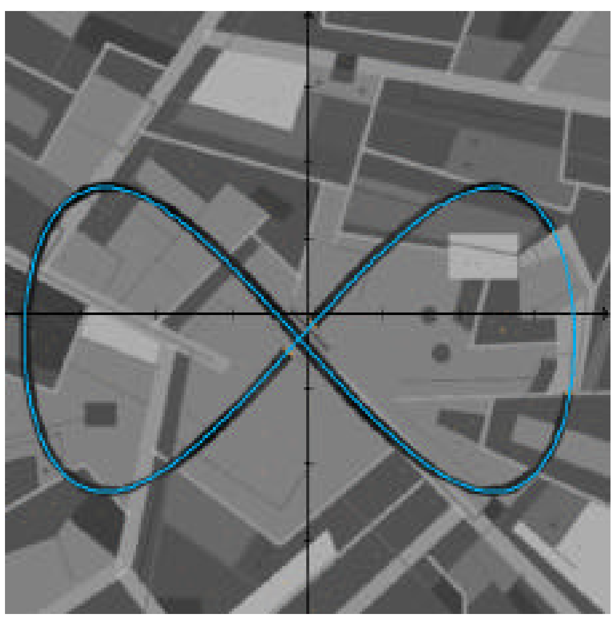

Figure 12 The Infinity City Skate Board Park

The Limaçon Pool at the High Up Resort

As an energy conservation measure, the management at the High Up Resort wants to order a specially made thermal cover for their limaçon pool. The material used in the cover is very expensive. To minimize their cost of the cover, they have hired you to accurately calculate the surface area of their pool. You are required to find the polar equation that best models the shape of the pool and use integration to find the surface area of the pool. The units of the x- and y-axes are meters.

The learning objectives for this game are:

1. Becoming familiar with a variety of polar equations.
2. Developing an understanding for the role the parameters play in shaping the graph of a polar equation.

The polar equation that best models the shape of the pool is

$$r = 15 + 9\cos(t)$$

Figure 13 The Limaçon Pool at the High Up Resort

Playing Soccer on a Slope Field

The objective of this soccer game is to find an initial value of y at $t = 0$ that will score a goal using the solution curve to the selected differential equation. Use the plotted slope field to determine where you should place the initial value. If you kick a goal on your first try, you score 10 points. If you kick a goal on your second try, you score 9 points. Thus, you get one less point for each try. After ten tries, you get zero points. The game consists of four quarters. In each quarter, you select one of the hidden differential equations in TEMATH's Work window, plot the slope field, and kick a goal. Your final score is the total points awarded for all four quarters.

The learning objective for this game is:

1. Drawing a solution curve through a slope field.

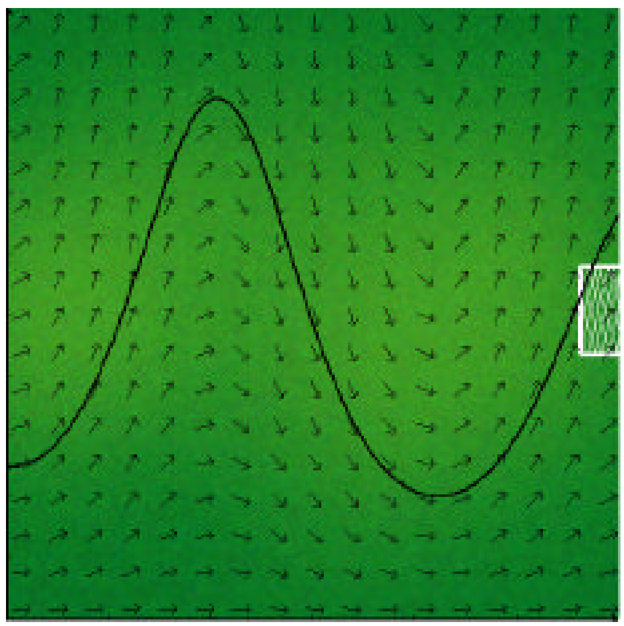

Figure 14 Playing Soccer on a Slope Field

Turkey Shoot with Trigonometric Functions — A Special Game for the Thanksgiving Holiday

> With Thanksgiving a few days away, the turkeys are out on the hunt. With their muskets programmed to fire periodic functions, they're in search of roaming Pilgrims. As the number one turkey musket programmer, it is your task to find a trigonometric function whose graph begins at the muzzle of the turkey's musket and ends at the Pilgrim's hand. Your pay for this job depends on the accuracy of your trigonometric function. You'll be paid $1000 if your first trig function hits its mark, $900 if your second trig function hits its mark, and so on. If you miss in ten tries, you're fired.

The learning objectives for this game are:

1. Building a library of trigonometric functions.
2. Understanding amplitude, frequency, and shifting.

An example equation of a trigonometric shot is

$$f(x) = 2.2\cos(1.2(x + 3.7)) + 0.2, \quad -3.72 \le x \le 3.8$$

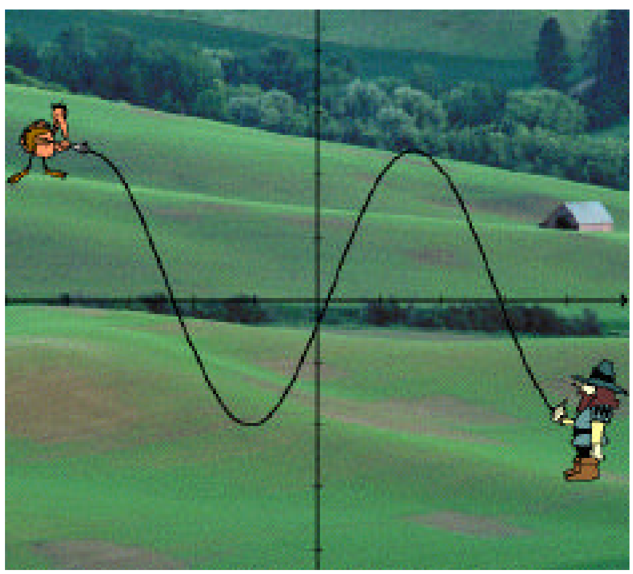

Figure 15 Turkey Shoot with Trigonometric Functions